WHY TRAINS STAY ON TRACKS

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Abstract

In this Article we give a qualitative and quantitative explanation of why a train stays on the rails in spite of perturbations which could make the wheels lurch and eventually derail it. We show that the stability originates in the conical shape of the wheels – a lateral disturbance gives rise to an asymmetric normal reaction from the two rails and a resultant restoring force. We first demonstrate translational stabilization in a simple situation where the rails are assumed frictionless and the steering motion of the wheel is neglected. Subsequently we develop a more comprehensive model, taking this motion into account. It is seen that rolling friction couples the rotational motion to the translational one and the original stability gets extended to include both cases. We find approximate formulae for the parameters governing stability and show that these are satisfied by a real railway coach to a high degree of accuracy.

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Introduction

'Why trains stay on tracks?' One takes this phenomenon for granted but there is no reason why a kilometre long train should remain on two narrow rails a metre and a half apart. Occasionally trains do derail with serious and sometimes tragic outcomes. A clear understanding of why trains normally do *not* derail should help in understanding why they do and may help in preventing derailments. This paper provides an explanation of what keeps trains from flying off the rails.

General interest in the problem of railway coach stability was aroused by a video of RICHARD FEYNMAN [1] where he debunks the flange theory (Fig. 1 left panel) and discusses the conical shape of the wheels (right panel); he mentions a translation-rotation coupling in the wheel as the primary stabilizing agent. This coupling is driven by rolling without slipping, and hence by frictional forces. The same explanation can also be found in the engineering literature [2-10], where it is known as the kinematic oscillation of J KLINGEL. In a 1965 paper, A H WICKENS [4] attributes the stability to the normal reaction rather than friction; in a 1998 publication [7] however WICKENS again mentions the friction-driven kinematic oscillation as the source of stability.

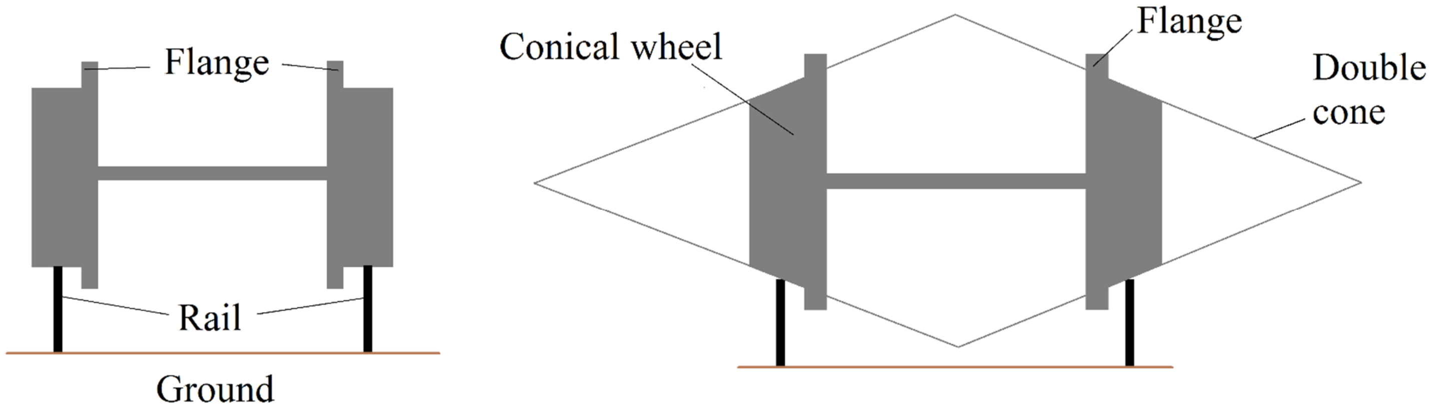

Figure 1 : *Various kinds of railway wheels, all in back view. Left panel shows flanges on cylindrical wheels – a popular misconception. Right panel shows a schematic of the actual wheel – the profile is conical. In this and subsequent figures, the conical angle is exaggerated for clarity – a typical realistic value is 3°.*

In this Article we show that while the translation-rotation coupling definitely exists, the dynamics of the reduced system is stable even if its strength is zero. The primary stabilizing agent is in fact the normal reaction. Using the dynamic model developed here, we obtain a dimensionless number, depending on the coach mass, size, moment of inertia etc. whose value acts as a measure of the coach stability. We show that for a real coach, this parameter comes out very close to its maximally stable value. The problem of train stability does not appear in standard texts on classical mechanics [11,12]; the only textbook where it features is the elementary [13], which includes a qualitative description of the rolling without slipping explanation and mentions an interesting toy demonstration of the phenomenon involving plastic cups and metre scales. In our Article, the level of presentation is quite suitable for the undergraduate or graduate classical mechanics curriculum.

1 Motion in the absence of friction

We consider a single axle as in the right panel of Fig. 1, neglecting the flanges as they are only a last resort mechanism. This is of course an idealized model of a railway coach but it captures the physics at the heart of coach stability. Further complications, some of which we will mention briefly in Section 3, result primarily in changes to numerical values of parameters characterizing stability. Thus, they are of great relevance in the design of trains but not really suitable as an exercise in an undergraduate or graduate classical mechanics course. The axle is conceptually replaced by a rigid double cone (shown in the Figure) which moves on horizontal rails (assumed ideal), making contact with them at two points. The initial state (whose stability we want to examine) features the cone (hereafter 'double' implicit) moving forward $(-\hat{\mathbf{y}})$ sitting symmetrically between the two rails and facing dead forward. An isometric view of the cone is shown in Fig. 2. The corresponding orthographic projections are in Fig. 3, which also introduces the definitions of the axes. Two perturbations on this reference configuration are of interest – a translational perturbation which is the x-coordinate x_{CM} of the cone centre of mass (CM), and a steering perturbation which is the yaw angle φ made by the cone with the forward direction. The relevant constants for the

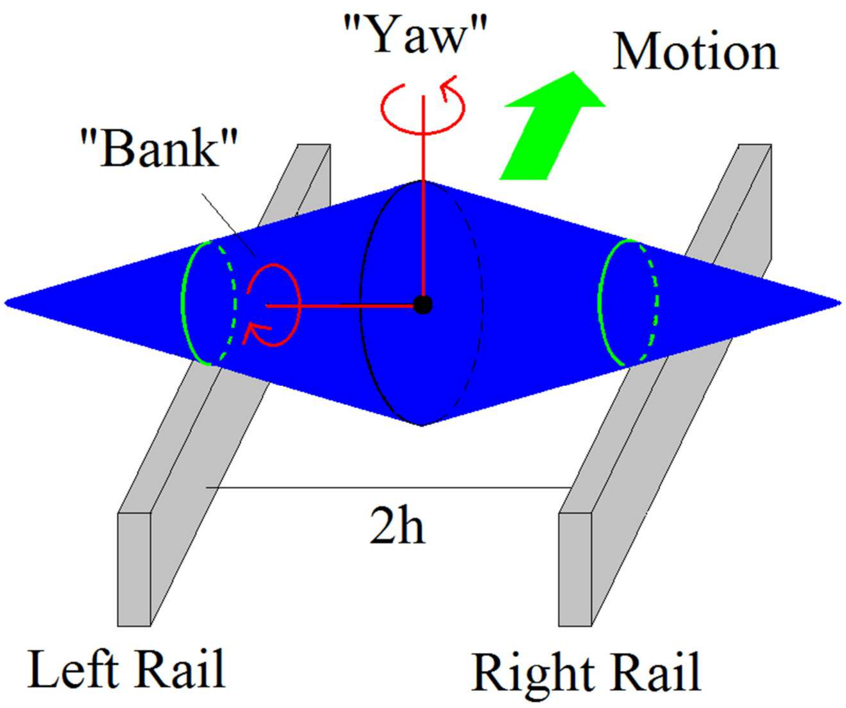

Figure 2 : *Isometric view of the double cone rolling along rails. The view is from the right back side. The direction of motion is shown by the arrow. To characterize the orientation of the cone after perturbations, we will use Euler angles – θ will be the angle of "bank" (i.e. tilt from the horizontal, about an axis parallel to the rails) and φ the angle of "yaw" (about a vertical axis). The contact circles between the cone and the rails are also indicated. The rails are shown as having finite width only for clarity; the analysis will treat them as having negligible width.*

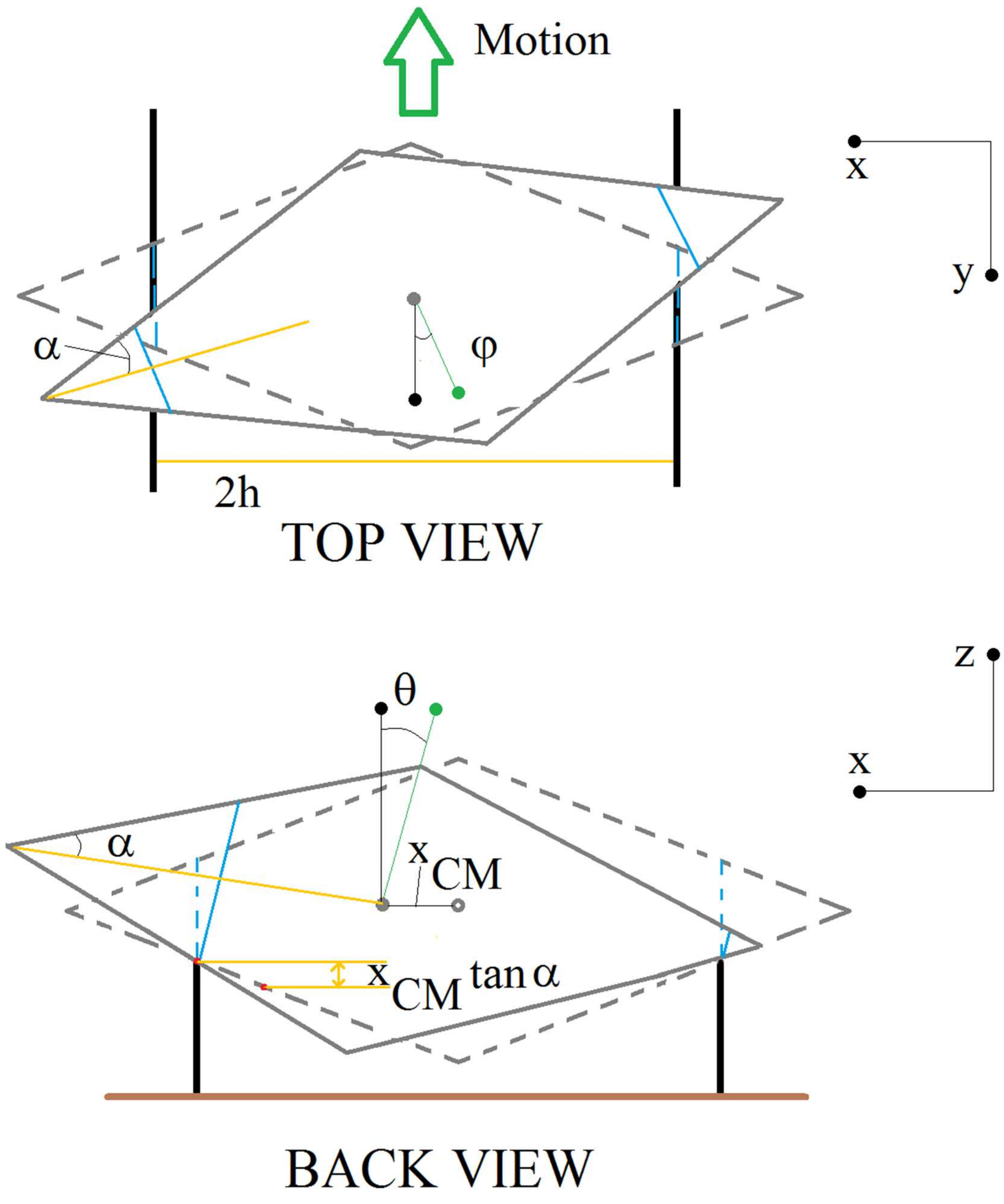

Figure 3 : *The perturbations x_{CM} (back view) and φ (top view). Solid and dashed lines denote new and reference configurations respectively. Blue lines indicate the circles of contact in the two configurations. TOP VIEW : Note that the contact circle is inclined after the rotation. BACK VIEW : Note that x_{CM} is positive and θ negative as per the axis conventions. Also, the radius of the contact circle increases at the left contact point and decreases at the right.*

problem are $2h$, the separation between the two rails, r_0, the radius of the cone-rail contact circle in the default state, α, the semi-vertical angle of the cone and the cone mass and its various moments of inertia.

To highlight the role of normal reaction we take the rails to be frictionless. In this case there cannot of course be any rolling without slipping. Hence, if the cone is found to be stable in this situation, it will prove convincingly that pure rolling is not an essential condition for train stability. Ignoring steering, we confine the analysis to the x-z plane, studying the response to a perturbation in the x-direction (as in the full problem). Like every two-dimensional mechanics problem, this has three degrees of freedom (DoF) : the x and z coordinates of the cone CM and the bank angle θ made in the plane about the y-axis. If we (realistically) take the mass of the cone to be huge, then there are also two constraints : the cone must remain in contact with *both* the rails at *all* times. Thus, the problem is 1DoF; we will engineer x_{CM} to be the chosen one.

The mathematical form of the actual constraint equations satisfied by the double cone is non-trivial, so we will use a physical argument to obtain a simplified but sufficiently accurate form. Starting from the reference configuration, suppose we perform the θ rotation alone keeping both x_{CM} and z_{CM} fixed at zero. Now if α is small (it is generally of the order of 0.05 radians) the cone looks almost like a needle. Then, for small positive θ, all points on the 'bottom of the cone' in the back view of Fig. 3 and lying to the left of the centre will go down by a distance $x\theta$ while all points on the cone bottom lying to the right of the centre will go up by distance $|-x\theta|$. In particular, the point at the original x-position of the left rail will move down $h\theta$ while the point at the original x-position of the right rail will move up $h\theta$. Now forget the rotation and suppose we perform a x-displacement of the CM alone with both other variables fixed at zero. Then, the point on the cone bottom at the x-position of left rail will move down $x_{CM}\tan\alpha$ while the point at the x-position of the right rail will move up $x_{CM}\tan\alpha$. Now if the rotation and translation are both applied, for a special relation between θ and x_{CM}, the upward and downward motions of the two points at the respective x-positions of the two rails will *both simultaneously cancel*, implying that the cone remains on both rails and satisfies both constraints. Since the z-displacement of the CM did not even enter the picture, it must be zero. Thus we have the relations :

$$h\theta = -x_{CM}\tan\alpha \Rightarrow \theta = -\frac{\tan\alpha}{h}x_{CM} \quad , \tag{1a}$$

$$z_{CM} = 0 \quad . \tag{1b}$$

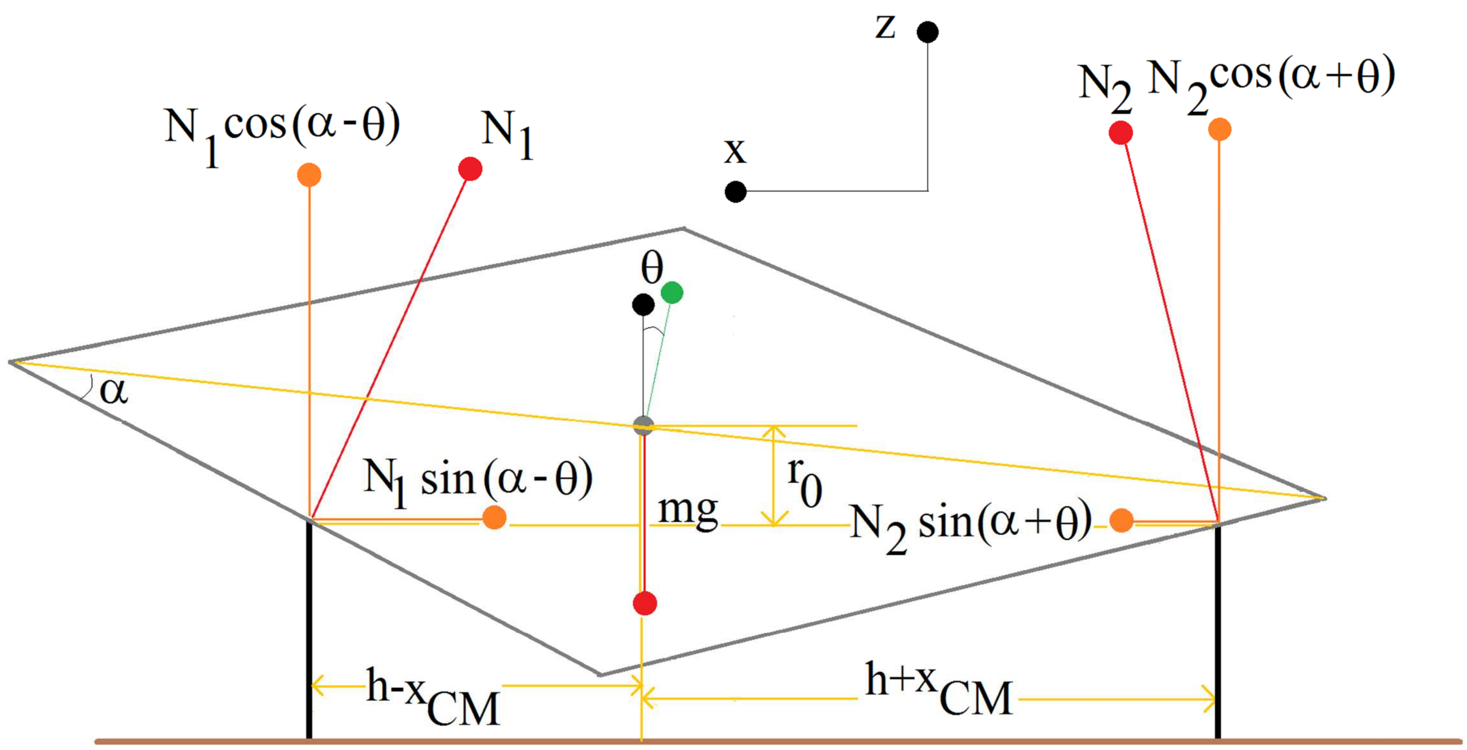

Figure 4 : *Free body diagram of the cone, in the x-z plane (back view). Red lines denote forces, orange their respective components. The forces are not to scale and labels indicate their magnitudes (the lines represent the directions). Yellow lines are for dimensioning. Recall that* θ *is* negative *for the configuration shown.*

We can now construct the free body diagram of the system, Fig. 4. The only forces acting on the cone are the normal reactions with magnitudes N_1 and N_2 from the left and right rails respectively. These reactions must be normal to the *cone* in addition to the rails – for the ideal rail considered here, *any* vector lying in the x-z plane qualifies as a bonafide normal. In terms of the reactions we can write the equations of motion, starting with the forms which are valid in the absence of constraints. The x force balance is

$$m\ddot{x}_{CM} = -N_1 \sin(\alpha-\theta) + N_2 \sin(\alpha+\theta) \quad , \tag{2}$$

where m is the mass of the double cone. The z force balance is

$$m\ddot{z}_{CM} = N_1 \cos(\alpha-\theta) + N_2 \cos(\alpha+\theta) - mg \quad . \tag{3}$$

The torque balance is trickier due to the possible presence of gyroscopic effects but we note that the value of I_{xx} involved is that of the axle and the wheels; on the other hand the value of I_{yy} involved in the θ rotation is that of the coach as a whole. These two moments of inertia are perhaps in the ratio of 10000:1 in favour of the coach and so the gyroscopic effect can be neglected. Then we can write the simple torque balance equation, assuming that y-axis is a principal axis for the cone and using $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ to calculate torque,

$$I_y \ddot{\theta} = r_0 \left[N_1 \sin(\alpha-\theta) - N_2 \sin(\alpha+\theta) \right] - (h - x_{CM}) N_1 \cos(\alpha-\theta) + (h + x_{CM}) N_2 \cos(\alpha+\theta) \quad . \tag{4}$$

Equations (2-4) are the equations of motion of the system.

From this point onwards it is just a few algebraic steps to the solution. We assume that the perturbations x_{CM} and θ are small, so that terms only up to linear order can be retained. This assumption is reasonable – the Grossmann Hartmann theorem guarantees that the linear behaviour is always accurate for sufficiently small perturbations, and for a train the perturbations had better not grow beyond the linear level if it is to run smoothly in practice. Further, the smallness of α means that its trigonometric functions can be linearized. The constraint $z_{CM}=0$ must be substituted into (3), and $\theta=-kx_{CM}$ where $k=(\tan\alpha)/h$ plugged into (4); the results after this are

$$N_1 + N_2 = mg \quad , \tag{5a}$$

$$m\ddot{x}_{CM} = -N_1 (\alpha + kx_{CM}) + N_2 (\alpha - kx_{CM}) \quad , \tag{5b}$$

$$-I_y k \ddot{x}_{CM} = -mr_0 \ddot{x}_{CM} - (h - x_{CM}) N_1 + (h + x_{CM}) N_2 \quad . \tag{5c}$$

These three equations can be used to eliminate the two reactions and get a single equation for $\ddot{x}_{CM}$; using the value of k that equation is

$$\left(\frac{I_y \alpha}{h} - mr_0 + \frac{mh}{\alpha} \right) \ddot{x}_{CM} = -2mgx_{CM} \quad . \tag{6}$$

This looks like a harmonic oscillator equation; indeed it is one because from the geometry r_0 is clearly less than h/α and hence the coefficient of $\ddot{x}_{CM}$ is always positive. Thus, if the double cone on the rails is given a lateral perturbation, then that perturbation remains bounded in time and hence the motion is stable. This analysis demonstrates incontrovertibly that the normal reaction by itself is sufficient to cause the railway wheel to remain stable in response to lateral disturbances.

2 Motion in the presence of friction

Although the normal reactions accounted for lateral stability, they are unable to generate steering (φ) stability. Since the normal reactions should be perpendicular to the rails, in the top view of Fig. 3 the reactions from both rails will be parallel to the x-axis. In the presence of a small steering perturbation φ about the z-axis, the two reactions would no longer be collinear, hence they would create a torque and amplify this perturbation, causing instability. To rescue this situation friction enters the picture, and we now need a quantitative treatment. To set up a framework consistent with the application of friction, we must be more specific in our definition of the initial reference state. We assume that this state features motion at a constant velocity v_0, i.e. the railway coach is traction-free, and that the double cone is rolling without slipping on the rails. This implies that the initial angular velocity of the cone must be $\omega_0 = v_0/r_0$ about the x-axis. With this step taken care of, we must mirror the steps leading to (6).

Assuming φ to be small, like all the other perturbations, the first observation is that the constraint equation (1) remains unchanged. This is because, after the φ rotation, the projections of all dimensions onto the x-z plane involve $\cos\varphi$ components, and their deviations from unity are of the second order of smallness. Our next observation features the normal reactions. Again referring to Fig. 4, the vertical (z) components of $\mathbf{N}_L$ and $\mathbf{N}_R$ (L: left R: right and a distinction has been drawn between the normal reaction vectors and their magnitudes N_1 and N_2 respectively) must remain as they were in the absence of φ rotation (this statement should in fact be true for *all* φ and not just *small* φ). At the level of accuracy of the calculation in this Article, these components are of size unity, and that does not change here. The horizontal components too remain as they were because of the requirement that they be in the plane perpendicular to the rails. Linearizing wherever appropriate, we have

$$\mathbf{N}_L = N_1\hat{\mathbf{z}} - N_1(\alpha-\theta)\hat{\mathbf{x}} \quad , \tag{7a}$$

$$\mathbf{N}_R = N_2\hat{\mathbf{z}} + N_2(\alpha+\theta)\hat{\mathbf{x}} \quad . \tag{7b}$$

Now comes the friction and this is where we will make a big assumption in lieu of a more accurate (and physically less transparent) calculation, as in the works of J J KALKER [14]. The ultimate purpose of the friction is to achieve rolling without slipping i.e. to ensure that the velocities of the contact points on the cone with respect to the ground are both zero. Thus, any relative velocity between contact point and rail will be opposed by the friction. Further, since the default state is traction-free by definition, we use the simplest friction law : a force at each contact point proportional to the velocity of that point with respect to the ground and opposite to this velocity in sign. This simplistic model of friction is no more drastic an approximation than the replacement of complicated dampings in vibrating machines by simple viscous dissipation – in both cases the plausibility of the final results is not affected by the substitution, and an astute choice of the damping strength gives quantitatively accurate results.

Working with this simple frictional form, we must calculate the velocities of both the contact points relative to the ground. This requires careful geometry : adding the various terms contributing to contact point velocity we get

$$\mathbf{v}_L = -v\hat{\mathbf{y}} + \dot{x}_{CM}\hat{\mathbf{x}} + \omega\left(r_0 + x_{CM}\tan\alpha\right)\hat{\mathbf{y}} - \varphi\omega\left(r_0 + x_{CM}\tan\alpha\right)\hat{\mathbf{x}} + \left(h - x_{CM}\right)\dot{\varphi}\hat{\mathbf{y}} \quad , \tag{8a}$$

$$\mathbf{v}_R = -v\hat{\mathbf{y}} + x_{CM}\hat{\mathbf{x}} + \omega\left(r_0 - x_{CM}\tan\alpha\right)\hat{\mathbf{y}} - \varphi\omega\left(r_0 - x_{CM}\tan\alpha\right)\hat{\mathbf{x}} - \left(h + x_{CM}\right)\dot{\varphi}\hat{\mathbf{y}} \quad . \tag{8b}$$

These expressions are complicated and we now go about the task of simplifying them.

The key observation is that v and ω will change from their initial values v_0 and ω_0 on account of forces and torques generated by the normal reaction and friction. However, since the train speed is controlled by the locomotive, a feedback between perturbation and speed cannot (and should not) enter the evolution equations for the perturbations – they must be stable independent of the existence of such a coupling. This motivates us to replace v and ω by v_0 and ω_0 in (8) and neglect all dynamics of these variables. Doing this substitution, using that $v_0=\omega_0 r_0$ and completely linearizing (8) we get

$$\mathbf{v}_L = \dot{x}_{CM}\hat{\mathbf{x}} + \omega_0\alpha x_{CM}\hat{\mathbf{y}} - \omega r_0\varphi\hat{\mathbf{x}} + h\dot{\varphi}\hat{\mathbf{y}} \quad , \tag{9a}$$

$$\mathbf{v}_R = \dot{x}_{CM}\hat{\mathbf{x}} - \omega_0\alpha x_{CM}\hat{\mathbf{y}} - \varphi\omega r_0\hat{\mathbf{x}} - h\dot{\varphi}\hat{\mathbf{y}} \quad . \tag{9b}$$

Note that the term αx_{CM}, though it appears quadratic, cannot be dropped because α is a small *constant* and not a small *variable*.

This is much simpler than (8) but we are not done yet. When we substitute these velocities into the friction constitutive relation, we will be getting some terms which are proportional to x_{CM} and φ and some which are proportional to their derivatives. Now, the whole thing will eventually enter the right hand sides of equations for $\ddot{x}_{CM}$ and $\ddot{\varphi}$, and in this kind of structure the stability is determined principally by the coefficients of the variables themselves rather than their first derivatives. The latter amount to damping terms by default while the former can constitute a harmonic oscillator or a harmonic repeller (imaginary frequency, exponential solutions) depending on the properties of the coefficients. Now if the system is an oscillator, the damping will cause the bounded solutions to behave even better and become zero in time; if the system is a repeller, the damping will not rescue the ill-behaved growing solutions. Hence the terms of primary interest in (9) are the ones featuring x_{CM} and φ rather than their derivatives, and we will write the frictional force taking only these into account. Now imposing the constitutive equation $\mathbf{f}=-\gamma\mathbf{v}$ where γ is the damping coefficient, we have

$$\mathbf{f}_L = -\gamma\alpha\omega_0 x_{CM}\hat{\mathbf{y}} + \gamma\omega_0 r_0\varphi\hat{\mathbf{x}} \quad , \tag{10a}$$

$$\mathbf{f}_R = \gamma\alpha\omega_0 x_{CM}\hat{\mathbf{y}} + \gamma\omega_0 r_0\varphi\hat{\mathbf{x}} \quad . \tag{10b}$$

Thus all the forces on the double cone have been determined.

The next step is the determination of the torques. This time both T_y and T_z will be relevant, the former being equal to $I_y\ddot{\theta}$ as before and the latter to $I_z\ddot{\varphi}$, where I_z is the moment of inertia about the z-axis (assumed principal). Since both these angular momenta are small, there is no question of gyroscopic coupling between the two. The position vectors from the centre of the cone to the left and right contact points are found from basic geometry and projection arguments similar to the preceding ones; they are

$$\mathbf{r}_L=(h-x_{CM})\hat{\mathbf{x}}+h\varphi\hat{\mathbf{y}}-r_0\hat{\mathbf{z}} \quad , \tag{11a}$$

$$\mathbf{r}_R=-(h+x_{CM})\hat{\mathbf{x}}-h\varphi\hat{\mathbf{y}}-r_0\hat{\mathbf{z}} \quad . \tag{11b}$$

Crossing these with the total forces (reaction and friction) at left and right contacts points gives the torques, and in terms of the resultant forces and torques we have the following equations of motion of the double cone :

$$m\ddot{z}_{CM}=N_1+N_2-mg \quad , \tag{12a}$$

$$m\ddot{x}_{CM}=-N_1(\alpha-\theta)+N_2(\alpha+\theta)+2\gamma\omega_0 r_0\varphi \quad , \tag{12b}$$

$$I_y\ddot{\theta}=-2\gamma\omega_0 r_0^2\varphi+r_0\left[N_1(\alpha-\theta)-N_2(\alpha+\theta)\right]-(h-x_{CM})N_1+(h+x_{CM})N_2 \quad , \tag{12c}$$

$$I_z\ddot{\varphi}=-2\gamma h\alpha\omega_0 x_{CM}+2(N_1+N_2)\alpha h\varphi \quad . \tag{12d}$$

It is now a simple matter to impose the constraints, eliminate the normal reactions and obtain a pair of coupled second order differential equations for x_{CM} and φ; they are

$$\left(\frac{I_y\alpha}{h}-mr_0+\frac{mh}{\alpha}\right)\ddot{x}_{CM}=-2mgx_{CM}+\frac{2\gamma h\omega_0 r_0}{\alpha}\varphi \quad , \tag{13a}$$

$$I_z\ddot{\varphi}=-2\gamma h\alpha\omega_0 x_{CM}+2mgh\alpha\varphi \quad . \tag{13b}$$

The substitution $Y=h\varphi$ now adds clarity to the presentation by making the above system dimensionally homogeneous; expressing it in a matrix form we have :

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{bmatrix}x_{CM}\\ Y\end{bmatrix}=-\begin{bmatrix}\dfrac{2mg}{\dfrac{I_y\alpha}{h}-mr_0+\dfrac{mh}{\alpha}} & \dfrac{-2\gamma\omega_0 r_0}{\alpha\left(\dfrac{I_y\alpha}{h}-mr_0+\dfrac{mh}{\alpha}\right)}\\ \dfrac{2\gamma h^2\alpha\omega_0}{I_z} & -\dfrac{2mgh\alpha}{I_z}\end{bmatrix}\begin{bmatrix}x_{CM}\\ Y\end{bmatrix} \quad . \tag{14}$$

This is the governing equation of the double cone in the presence of friction considering both lateral displacement and steering motions.

Equation (14) describes two coupled second order systems. What we want to do is express it in terms of new variables, $Z_1=ax_{CM}+bY$ and $Z_2=cx_{CM}+dY$ where a,b,c,d are constants, such that the equation acquires a structure $\ddot{Z}_1=-E_1Z_1$ and $\ddot{Z}_2=-E_2Z_2$. Now, if E_1 and E_2 are real and positive, then both are like harmonic oscillators where solutions remain bounded in time; in all other cases the system is a harmonic repeller where solutions blow up. Expressing (14) as

$$\ddot{\mathbf{X}}=-\mathbf{K}\mathbf{X} \quad , \tag{15}$$

we now invoke a theorem in linear algebra which says that this can be written as

$$\begin{bmatrix}Z_1\\ Z_2\end{bmatrix}=\mathbf{PX} \quad , \qquad \mathbf{P}\ddot{\mathbf{X}}=-\mathbf{D}(\mathbf{PX}) \tag{16}$$

where $\mathbf{D}$ is a diagonal matrix with elements E_1 and E_2 which are the eigenvalues of $\mathbf{K}$ and $\mathbf{P}$ is a matrix whose columns are the eigenvectors of $\mathbf{K}$. This has an uncoupled structure, and for stability we want E_1 and E_2 to be both real and positive. (We note that this diagonalization is exactly what is done to find the normal modes of a coupled spring mass system.)

Using the standard formula for the eigenvalues in terms of the trace Tr and the determinant Det of $\mathbf{K}$, we have

$$E_{1,2}=\frac{Tr\pm\left(Tr^2-4Det\right)^{1/2}}{2} \quad , \tag{17}$$

If Det is negative, then the surd will have greater magnitude than Tr and one of the eigenvalues will come out negative. When Det is increased to zero, the negative eigenvalue just becomes zero. As Det turns positive, the surd becomes smaller in magnitude than Tr and hence both roots come out positive, which is what we want. This regime will not last indefinitely however. As we continue increasing *Det*, the surd will eventually become zero when *Det* equals 1/4 the square of the trace; any further increase in *Det* will make the eigenvalues imaginary which is again a useless operating regime. These considerations indicate that the driving parameter is in fact a dimensionless number, the ratio of the determinant of $\mathbf{K}$ to the square of its trace. We call this ratio λ. Since operation is stable for λ lying between 0 and 1/4, an optimally stable railway coach should logically have

$$\lambda = \frac{Det}{Tr^2} = \frac{1}{8} \quad , \tag{18}$$

as far away as possible from either region of instability. Within the stable region, the primary source of stability is K_{11}, which arises from the normal reaction (this quantity had also appeared in the frictionless analysis and it is independent of γ), and the frictional coupling splits this stability among both modes of motion. Thus it is indeed the normal reaction and not friction which is primarily responsible for the train's stability. The ratio λ is an indicator of the strength of the coupling relative to the strength of the self-stabilizing action – if it is too weak, the steering mode is unstable while if it is too strong, it overcomes the restoring tendency of the reaction and drags both modes of motion into instability.

3 Discussion of approximations and comparison with reality

Our model has involved considerable approximations, foremost among them being the simplified friction model and the restriction to a single-axle treatment ignoring the effect of the bogies. The friction model is qualitatively correct because it opposes relative motion between wheel and rail, which is what any friction would always do. The nature of this opposition would in reality be different from our simple formula (as in Reference [14]), but the direction of the force – restoring or augmenting – which is what ultimately determines stability, would remain unchanged.

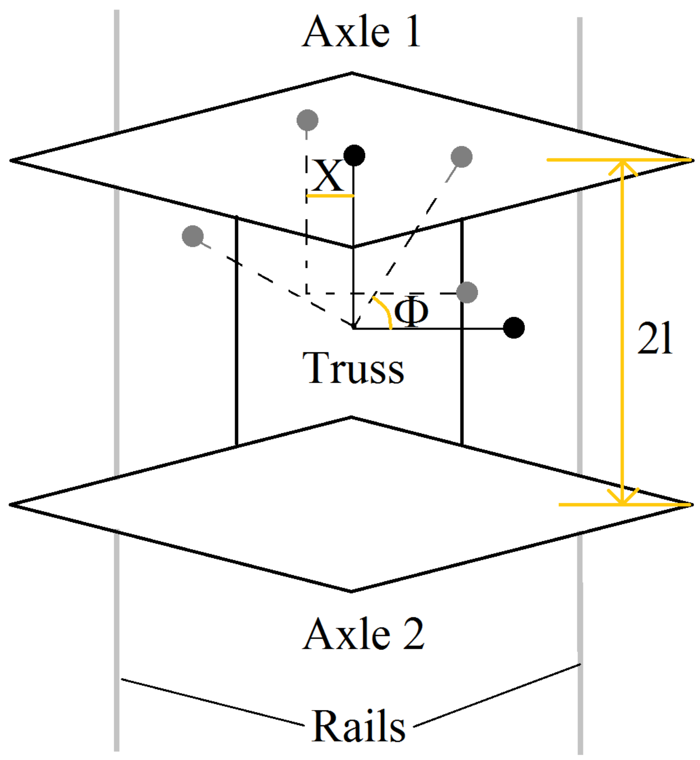

Figure 5 : *Top view of a bogie. The two axles are modelled as double cones and they are connected by a rigid truss. The black solid lines indicate a bogie-fixed axis. The dotted lines show how their position and orientations change for each of the two bogie perturbations considered here.*

As for the bogie issue, we consider a rigid bogie with two identical axles attached to the ends, shown in Fig. 5. Once again, there are two modes of motion in this system – translation X and steering Φ of the bogie as a whole. Since the system is linear, the response to a combination of translation and steering can be written as a sum of the two individual responses. First, suppose the bogie is displaced to the left with no steering. Then, each axle looks like that of Fig. 4 and as per (13a) experiences a restoring force, which is proportional to X through a negative constant $-P$. Since this force is the same for both axles, they do not generate a moment about the bogie centre of mass. As per

(13b), the displacement also creates a torque on each axle, which is proportional to X through another negative constant $-R$. Since the bogie is rigid, the torques of the two axles add and the resultant torque goes into the equation for $\ddot{\Phi}$. Thus, in the presence of a displacement perturbation alone, we have a structure like

$$\begin{aligned}\ddot{X} &= -(2P/M)X \\ \ddot{\Phi} &= -(2R/J)X\end{aligned}\quad , \tag{19}$$

where M and J are the mass and moment of inertia of the bogie respectively. Now suppose that the bogie is given a steering perturbation Φ with no displacement. Then the two axles experience a translational displacement by $\pm l\Phi$ where $2l$ is the length of the bogie. Again, from (13a), the force on each axle on account of this displacement is $\mp Pl\Phi$; since they are equal and opposite for the two axles, they cancel and the net force on the bogie is zero. These forces do have a finite torque however, and it opposes the displacement; from the geometry, its value is $-2Pl^2\Phi$. Over and above this, (13a) says that there is a force on each axle due to its steering perturbation; this is proportional to Φ through positive constant Q, and total force on the bogie is the sum of the forces on the axles. Finally, from (13b) the steering perturbation Φ on each axle gives rise to a torque proportional to Φ through some positive constant S. Adding all these,

$$\begin{aligned}\ddot{X} &= (2Q/m)\Phi \\ \ddot{\Phi} &= (1/J)\left(-2Pl^2 + 2S\right)\Phi\end{aligned}\quad . \tag{20}$$

The two equations can now be combined into a matrix :

$$\begin{bmatrix}\ddot{X}\\ \ddot{\Phi}\end{bmatrix} = -2\begin{bmatrix}P/m & -Q/m\\ R/J & \left(Pl^2 - S\right)/J\end{bmatrix}\begin{bmatrix}X\\ \Phi\end{bmatrix} \quad . \tag{21}$$

Again, the system will be stable if both the eigenvalues of the matrix above are positive and real. This matrix is clearly related to the one in (14); if that is stable, this too is likely to be so, unless the bogie dimensions are designed pathologically. Analogous calculations can yield results for bogies consisting of 3, 4 or more axles (common in erstwhile steam locomotives). The main thing to note is that the basic conclusions are not substantially affected by the presence of bogies. In addition to this, bogies pose definite advantages when stability to pitching motions (rotation about a horizontal axis perpendicular to the rails) is considered [15] but the treatment of these will take us too far afield.

To validate the simplifications and approximations which we have made, we now perform a comparison of our results (14-18) with values from a realistic coach. If the agreement is good, then our model is valid while if the agreement is poor, then we will need refinements. An LHB coach used on Rajdhani Express trains of Indian Railways and shown in Fig. 6, has the following dimensions and weights [16]:

Figure 6 : *LHB coach used on Rajdhani Express. This image is taken from Reference [15].*

- mass 45 tonnes
- length 24 m

- width 3.2 m
- height 4.0 m
- wheel radius 0.42 m
- track gauge 1.68 m
- cone angle 0.05 rad

Using these values and assuming the mass to be uniformly distributed throughout the coach, we get I_y=278000 kgm^2 and I_z=2198000 kgm^2. As a ballpark value of γ, we note that generally a train has a peak adhesion factor of 40% and we assume that this maximum friction occurs at a typical operating speed of 20 m/s (about 70 km/hr), which lies halfway between zero speed and the maximum permissible speed of the coach; this gives γ=10000 N/(m/s) approximately. Plugging these numbers into (14) yields the eigenvalues 1.0 and 0.18 s^{-2} which are both positive, implying that Rajdhani Express trains are stable (as we indeed know they are). It is interesting to now compute the ratio λ – for the LHB coach this actually evaluates to 0.124, very close to the optimal value of 1/8 obtained above.

Now in a realistic situation, the friction coefficient is a given and the dimensions and masses too are to a large degree invariant from coach to coach. The real variable is the cone angle – in the early days of Indian Railways this was fixed at 1 in 20 (0.05 radians) by a British-derived convention based on extensive experimental trials. Our simple theory is able to vindicate this choice as being optimally stable. In Fig. 7 we present a graph of the ratio λ as the angle α is varied while all other parameters are held constant. The numerical values are as for the LHB coach. It is seen that λ equals 1/8 at a value of α very close to 0.05. For design and construction of new high speed train sets etc. where parameter values can be significantly different from a conventional setup, the criterion (18) may be taken as a starting point to evaluate stability of the train.

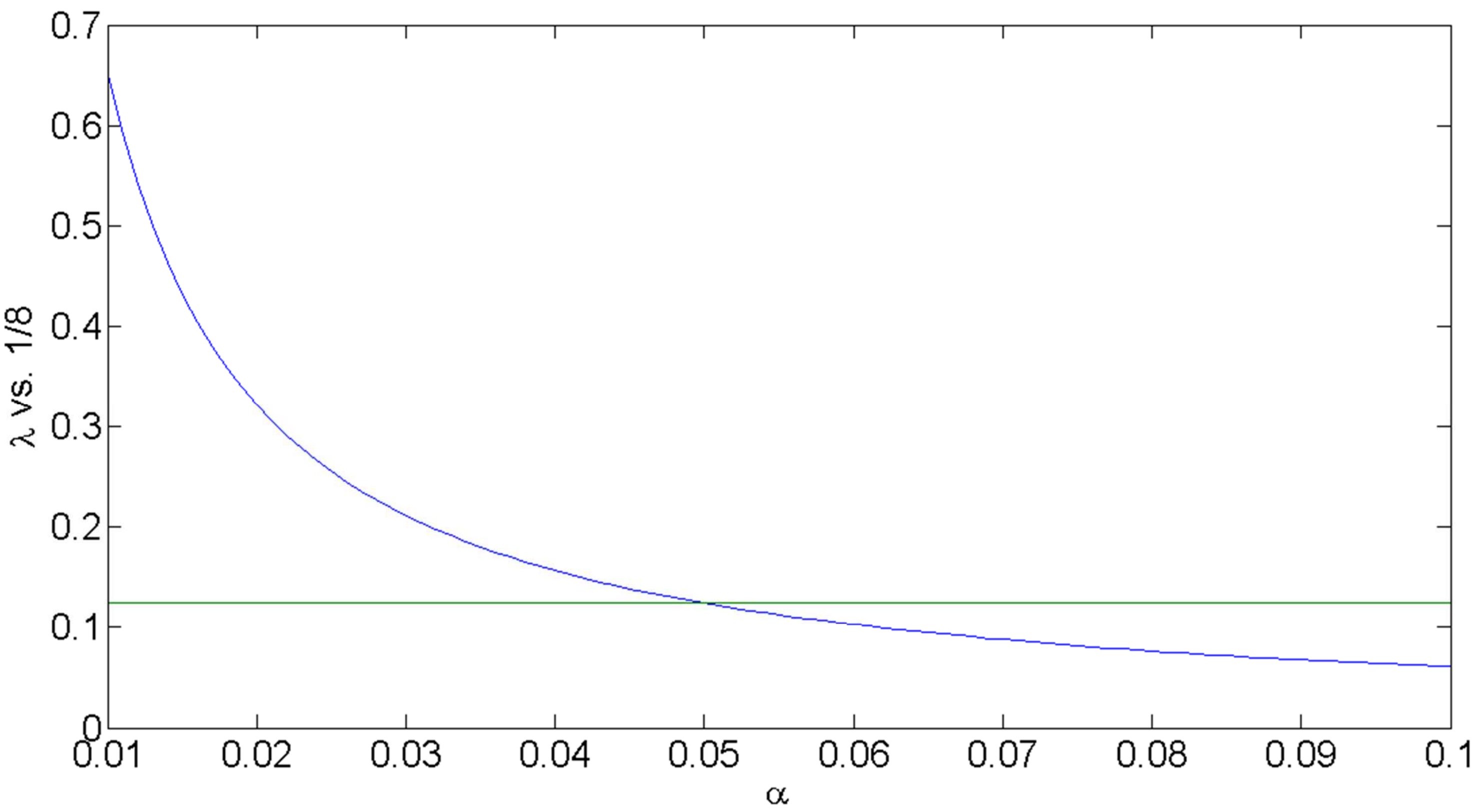

Figure 7 : *The blue line shows λ vs α while green is the constant quantity 1/8. The two curves intersect very close to α=0.05.*

This strong numerical agreement bolsters our argument nicely.

Finally we make a comment on the applicability of Grossmann Hartmann theorem which we invoked to justify the linearized analysis. The theorem is valid only if the eigenvalues have nonzero real parts. However it appears from (14) that both eigenvalues here are purely imaginary, thereby nullifying the theorem application. This apparent contradiction is resolved by noting that (14) was obtained by specifically dropping the damping terms from (9) and retaining only the position-dependent terms. In reality, both the eigenvalues will acquire negative real parts from the damping superposed on the imaginary components from (14) and thus the application of Grossmann Hartmann theorem remains valid.

In conclusion we would like to highlight a comment by WICKENS [7], which is that nobody knows who invented this mechanism. It is strange indeed that the identity of the inventor of coned wheels, without which trains would not have existed, is shrouded in anonymity.

* * * * *

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